L. *A Supplement to a former Paper, concerning Difficulties in the* Newtonian *Theory of Light: by the Rev.* S. Horſley, *L L. B. F. R. S.*

PROBLEM I.

Read Dec. 19, 1771.

A PARCEL of equal Circles being diſpoſed upon a plane ſurface, of any figure whatſoever, in the cloſeſt arrangement poſſible, to determine the ultimate proportion of the ſpace covered by all the Circles, to the ſpace occupied by all their Interſtices, when each circle is infinitely ſmall, and the ſpace, over which they are diſpoſed, is of a finite magnitude.

The cloſeſt manner, in which a parcel of equal circles can be diſpoſed upon a plane, is when the centers of every three contiguous circles are ſituated at the angles of an equilateral triangle, which hath each of its ſides equal to a diameter of any one of the circles.

A number of circles, thus diſpoſed, may be divided, as TAB. XVII. ſhews, into ſeveral rows of circles, having their centers ranged upon parallel right lines, A G, H P, Q X, Γ Σ, &c. Every

circle,

circle, which is not in an outermoſt row, or at the extremity of any other row, touches ſix others, namely two in its own row, and two in the row on either ſide of its own: and each adjacent pair of theſe ſix do alſo touch each other. In the outer rows, every circle, which is not at one extremity of its row, touches four others, two in its own row, and two in the row next beſide it; which laſt two do likewiſe touch each other. A circle at either extremity of an outer row, touches only a ſingle circle in its own row, but either one or two in the row next beſide it. The bare inſpection of the figure (TAB. XVII.) will make theſe aſſertions manifeſt.

Now, imagine the equal circles, exhibited in the figure, to be each infinitely ſmall, the number of them being infinitely great, and the whole ſpace over which they are diſpoſed being of a finite magnitude. The ultimate proportion of the ſpace covered by all the Circles, to the ſpace occupied by all their Interſtices, is that of $\frac{1}{2}$ the area of one of the circles to the whole of one interſtitial area, *i. e.* the proportion of 39 to 4 very nearly.

DEMONSTRATION.

The circles ranged along the parallel right lines AB, HP, form two rows of interſtices; the row marked *a*, *b*, *c*, *d*, &c. and the row marked α, β, γ, δ, &c. and, in like manner, two rows of interſtices are formed by every two contiguous rows of circles.

Now, the numbers of the circles ranged along the ſeveral parallel right lines, AG, HP, QX, &c. are either equal or unequal, according to the figure of the ſpace over which they are diſpoſed.

 Caſe I.

Caſe 1. Firſt ſuppoſe, that an equal number of circles is ranged along each of the parallel lines; in which caſe, the figure, in which they are included, muſt be a parallelogram. The number of circles, ranged along the parallel right lines AG, HP, being equal, the number of interſtices in each of the rows, *a*, *b*, *c*, *d*, &c. α, β, γ, δ, &c. is leſs by unity than the number of circles upon either line, AG, or HP, be that number what it will. Thus the two circles A, B, upon the line AG, with the two circles H, K, upon the line HP, have the ſingle interſtice *a*, in the row *a*, *b*, *c*, *d*, &c. and the ſingle interſtice α, in the row α, β, γ, δ, &c. Again, the three circles A, B, C, upon the line AG, with the three, H, K, L, upon the line HP, have the two interſtices *a*, *b*, in the row *a*, *b*, *c*, *d*, &c. and the two α, β, in the row α, β, γ, δ, &c. And univerſally, if the number of circles in each row be m, the number of interſtices, in each of the two rows of interſtices, will be $m - 1$. Conſequently, the whole number of interſtices formed by theſe two rows of circles is $2m - 2$. In like manner, the two rows of circles HP, QX, form two more rows of interſtices. And the number of circles upon each line, HP, QX, being m, the number of interſtices in each row is $m - 1$, and the whole number in both rows $2m - 2$, Therefore, the whole number of interſtices formed by the three rows of circles, AG, HP, QX, is $2m - 2$ twice taken, or $\overline{2m-2} \times 2$. By the ſame reaſoning if a fourth row of m circles, ΓΣ be added, the number of interſtices formed by the four rows is $\overline{2m-2} \times 3$. And univerſally, if there be n rows

of

of equal circles, and m circles in each row, the number of interſtices formed by all the rows is $\overline{2m-2} \times \overline{n-1}$. Now, when the circles are infinitely ſmall, their diameters are infinitely ſmall. Therefore, the ſpace which they cover being of finite magnitude, it is neceſſary, that both the number of circles in each row, and the number of rows, that is, that each of the numbers, m and n, ſhould be infinitely great. But when m and n are each infinitely great, $\overline{2m-2} \times \overline{n-1}$, that is, the number of interſtices, becomes ultimately $2mn$; and the interſtices being all equal one to another, if the area of one be called P, the ſum of their areas will be $2mn \times P$. But the number of circles in n rows, each row conſiſting of m circles, is mn; and the circles being equal, if the area of one be called A, the ſum of their areas will be $mn \times A$. Hence the ſpace covered by all the circles is to the ſpace covered by all their interſtices, when the magnitude of each circle is infinitely diminiſhed, and the number of them ſo infinitely augmented, as that they ſhall cover a ſpace of finite magnitude, ultimately, as $mn \times A$ to $2mn \times P$, that is, as A to $2P$, or as $\frac{1}{2}A$ to P, that is, as $\frac{1}{2}$ the area of one circle to the whole area of one interſtice.

Caſe 2. Now, ſuppoſe, that unequal numbers of circles are ranged along the ſeveral lines AG, HP, QX, &c. which muſt always be the caſe, if the figure of the ſpace, in which they are contained, be any other than a parallelogram; and let the number upon AG be the greateſt of all, and call that number, as before, m. If from the row HP, the extreme

treme circle P be taken away, all the reſt being left, the interſtice ζ will be taken away, and all the other interſtices, formed by m circles upon HP, with m circles upon AG, will remain. If again the circle O be taken away, beſides the interſtice ζ already taken away, the two ε, f will diſappear; and every circle more that is taken away, of thoſe remaining upon HP, from the extremity of the line, two more interſtices will diſappear. If from the row of circles HP, the extreme circle H be taken away, the two interſtices a, α, will diſappear. And if the circles K, L, M, be taken away ſucceſſively, every new circle that is taken away, two more interſtices will diſappear, of thoſe formed by the two rows AG, HP. Again, if the two circles P and H be taken away, the three interſtices ζ, a, α, will diſappear; and every circle more that is taken away, from either extremity, two more interſtices will diſappear. Hence whatever number of circles be taken away out of m circles upon HP, provided they be taken ſucceſſively, from either or both ends of the row (and when the number of circles upon HP is ſuppoſed leſs than that upon AG, the deficiency muſt be at the end, not in the middle of the row, otherwiſe the circles remaining would not be in the cloſeſt arrangement), it is evident that the number of interſtices which diſappear, of thoſe which would be formed by m circles upon HP, with m circles upon AG, muſt be either double the number of circles taken away, or leſs than the double of that number by 1. That is, if $m-a$ be the number of circles left upon HP, the number of interſtices formed by them, with m circles upon AG, is leſs than the number which would be formed by m circles upon HP, with m circles upon AG, either by $2a$, or by $2a - 1$. The number of interstices formed by m circles upon each row would be, as hath been shewn in the preceding case, $2m - 2$. Therefore, the number formed by m circles upon AG, with $m - a$ circles upon HP, is either $2m - 2 - 2a$, or $2m - 2a - 1$. That is, ultimately (when the number $m - a$ is infinitely increased) $2m - 2a$. Now, suppose the number of circles upon QX to be $m - a - b$. The number of circles upon the two rows AG, HP, is $2m - a$. Upon the three rows AG, HP, QX, the number is $3m - 2a - b$. And if the number of circles upon ΓΣ be $m - a - b - c$, the number of circles upon the four rows AG, HP, QX, ΓΣ, will be $4m - 3a - 2b - c$. And, universally, the number of rows being n, and the number of circles upon the several rows, m, $m - a$, $m - a - b$, $m - a - b - c$, $m - a - b - c - d$, &c. successively, the whole number upon all the n rows will be

$nm - a \times \overline{n - 1} - b \times \overline{n - 2} - c \times \overline{n - 3}$, &c.

But, as it hath been shewn that m circles upon AG, with $m - a$ circles upon HP, form $2m - 2a$ interstices, if the number $m - a$ be infinite, in the same manner it may be shewn, that $m - a$ circles upon HP, with $m - a - b$ circles upon QX, form $2m - 2a - 2b$ interstices, when the number $m - a - b$ is infinite. Therefore, the whole number of interstices formed by the three rows upon AG, HP, QX, is $\overline{2m - 2a} \times 2 - 2b$. And, in like manner, the number of interstices, formed by the circles of four rows, will be $\overline{2m - 2a} \times 3 - \overline{2b} \times 2 - 2c$. And, universally,

n being

n being the number of rows, the number of the interſtices will be

$\overline{2m-2a}\times\overline{n-1}-2b\times\overline{n-2}-2c\times\overline{n-3}$, &c. That is, $2m\times\overline{n-1}-2a\times\overline{n-1}-2b\times\overline{n-2}-2c\times\overline{n-3}$, &c. By comparing this expreſſion with the former expreſſion of the number of the circles, it will appear, that when n, the number of the rows of circles, is infinitely augmented, the number of interſtices is to the number of circles, ultimately, as 2 to 1. For the two expreſſions always conſiſt of an equal numbers of terms. The ſame numerical terms in both are affected with the ſame ſigns. The firſt term of the latter ($2m\times\overline{n-1}$) is ultimately double the firſt term of the former (mn), when n is infinitely increaſed, and each ſucceeding term of the latter is double the correſponding term of the former. Therefore, the whole of the latter expreſſion is ultimately to the whole of the former, as 2 to 1. That is, the number of interſtices is ultimately double the number of circles: whence it follows, as in the former caſe, that the whole ſpace covered by the circles is to the whole ſpace occupied by the interſtices, as $\frac{1}{2}$ the area of one circle to the whole area of one interſtice.

In this Demonſtration I have ſuppoſed the number of circles upon the ſeveral lines AG, HP, QX, &c. to decreaſe continually. Had I ſuppoſed them to decreaſe by fits, and in any manner imaginable, ſtill the concluſion would have been the ſame (a). There-

(a) Suppoſe the number of circles upon the 1ſt row to be m, upon the 2d, $m-a$, upon the 3d, $m-a+b$, upon the 4th, $m-a+b-c$, upon the 5th, $m-a+b-c+d$, and ſo on, and

 fore,

fore, let the figure of the finite ſpace, including the circles thus cloſely arranged, with their interſtices, be what it will, the proportion of the ſpace covered by all the circles, to the ſpace taken up in interſtice, is ultimately that of ½ the area of one circle to the whole area of one interſtice.

Now, that this is the proportion of 39 to 4, very nearly, will appear by computing one of the interſtitial areas.

The method of computing the interſtitial area is obvious. Let A, B, H be the centers of the three circles, which cloſe the interſtice Υ Φ Ψ. Join A B, A H, B H. The right lines A B, A H, B H, paſs through the points of contact Υ, Φ, Ψ, reſpectively.

and each of theſe numbers to be infinitely increaſed. Then, n being the number of rows, the whole number of circles will be

$$nm - a \times \overline{n-1} + b \times \overline{n-2} - c \times \overline{n-3} + d \times \overline{n-4}, \text{ \&c.}$$

Number the interſtices formed by every two contiguous rows, and add them all together, and the whole number of interſtices will be found to be

$$2m \times \overline{n-1} - 2a \times \overline{n-1} + 2b \times \overline{n-3} - 2c \times \overline{n-3} + 2d \times \overline{n-5}, \text{ \&c.}$$

Now, by comparing theſe two expreſſions, it appears, that both conſiſt of the ſame number of terms: That the ſame numerical terms in order from the firſt, have the ſame ſigns: That the firſt term of the latter ($2m \times \overline{n-1}$) is ultimately the double of the firſt term of the former, when n is infinitely increaſed: That of the terms following the firſt, the negative terms of the latter are each double the correſponding negative terms of the former: and each poſitive term of the latter differs from the double of the correſponding poſitive term of the former. by a number which vaniſhes with reſpect to either of thoſe correſponding terms, when n becomes infinite. Therefore, when n becomes infinite, the whole of the latter expreſſion becomes the double of the whole of the former. Hence the concluſion is as before.

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The area of the triangle, AHB, is equal to the areas of the three ſectors ΑΥΦ, ΒΦΨ, ΗΨΥ, added to the interſtitial area ΥΦΨ. But the triangle AHB is equilateral. Therefore each of the ſectors ΑΥΦ, ΒΦΨ, ΗΨΥ is $\frac{1}{6}$ of the circle to which it belongs: and, the circles being equal, the three ſectors are equal to the half of any one of the circles. Therefore, the area of the triangle AHB is equal to $\frac{1}{2}$ the area of one circle (as of A) added to the interſtitial area ΥΦΨ. Therefore, from the area of the triangle AHB take $\frac{1}{2}$ the area of the circle A, and there will remain the interſtitial area ΥΦΨ.

Now, if the radius ΑΦ be put = 1, each ſide of the triangle AHB will be 2.

Therefore, the area of the triangle AHB will be . . . . } = 1,73205
But the radius being 1, $\frac{1}{2}$ the area of the circle A is . . . . } = 1,5708

The difference is 0,1612

And this is the interſtitial area ΥΦΨ, the half area of the circle A being 1,5708. Therefore, the ſemicircle is to the interſtice as 1,5708 to 0,1612, or as 9,74 to 1, or as 39 to 4, very nearly.

COROLLARY.

If a parcel of equal circles be ſo diſpoſed upon a plane ſurface of any figure whatſoever, that the centers of every three adjacent circles are ſituated at the angles of equal equilateral triangles, having ſides greater than the diameters of the circles, but greater in a finite proportion, the ultimate proportion of the ſpace

 covered

covered by all the circles to the ſpace occupied by all the interſtices, when each circle is infinitely diminiſhed, and the number of them ſo infinitely increaſed, that the ſpace over which they ſpread is of a finite magnitude, is that of $\frac{1}{2}$ the area of one circle to the whole area of one interſtice. And the area of any one interſtice is equal to the difference of the area of the equilateral triangle, formed by the right lines joining three adjacent centers, and $\frac{1}{2}$ the area of one of the circles.

PROBLEM II.

To determine the greateſt poſſible denſity of an infinitely thin cruſt compoſed of equal ſpherules, having their centers all in the ſame plane.

From the number 39 ſubtract its third part. To the number 4 add the third part of 39. The remainder is to the ſum, that is, 26 is to 17, very nearly, as the ſpace occupied by all the matter to the ſpace occupied by all the pore, in an infinitely thin cruſt, of the greateſt poſſible denſity, compoſed of equal ſpherules, having all their centers in the ſame plane.

DEMONSTRATION.

Upon a baſe of innumerable infinitely ſmall circles, arranged in the cloſeſt manner poſſible, (according to Prob. I.) imagine right cylinders to be erected, each cylinder having one of the little circles for its baſe, and its altitude equal to the diameter of its baſe.

Theſe cylinders are in the cloſeſt arrangement poſſible for equal cylinders; and the ſpheres, which they circumſcribe, are in the cloſeſt arrangement poſſible for equal ſpheres, which have their centers in the ſame plane. The ſolid ſpace occupied by the cylinders, is to the ſolid ſpace occupied by their interſtices, as the ſurface covered by their circular baſes, to the ſurface covered by the interſtices of their baſes: That is, as 39 to 4, very nearly, by the firſt Problem. But the ſpheres contained within theſe cylinders are each but $\frac{2}{3}$ of the containing cylinder. The ſolid content therefore of all the ſpheres is but $\frac{2}{3}$ of the ſolid content of all the cylinders; and the remaining third part of the ſolid content of the cylinders, together with the interſtices between the cylinders, makes up the whole of the interſtices between the ſpheres. Therefore, the ſpace occupied by the ſpheres is to the ſpace occupied by their interſtices, as $39 - \frac{19}{3}$ to $4 + \frac{19}{3}$, or as 26 to 17, very nearly.

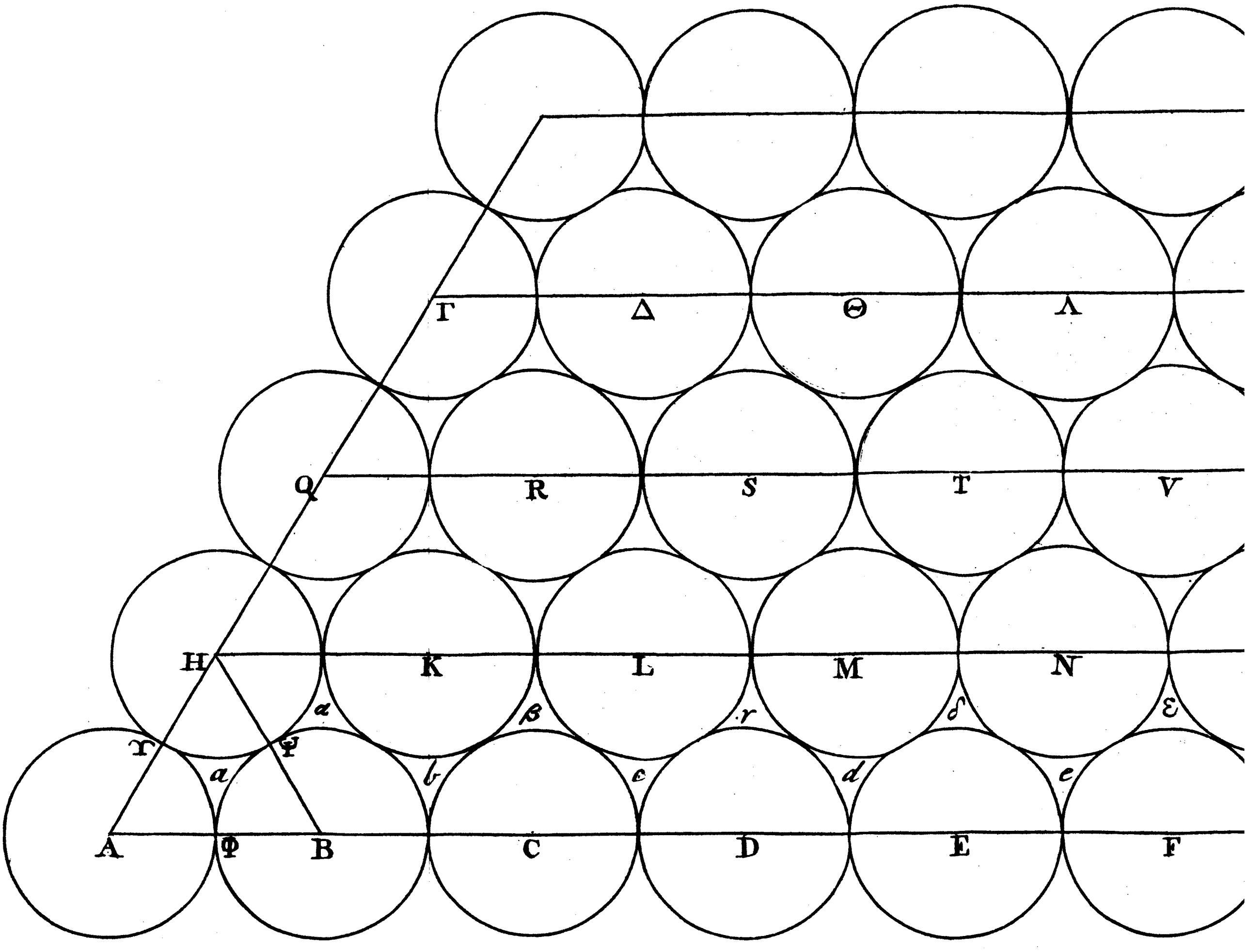
The ſpheres being in the cloſeſt arrangement poſſible, if each be a ſolid atom, or without pore within its own dimenſions, then, the infinitely thin cruſt, which theſe atoms compoſe, is plainly the moſt denſe that can be compoſed of equal ſpherules, having their centers in one plane. And the ſpace occupied by its matter is to the ſpace occupied by its pore, as 26 to 17, very nearly.

Scholium.

If the component ſpherules, inſtead of being ſolid, be ſuppoſed to be each of the denſity of gold, in which one half of the bulk may reaſonably be ſuppoſed to be be pore, then only $\frac{1}{2}$ of the ſpace, which they occupy, is filled with matter, and the other half is to be added to the pore. Hence ſpherules of the denſity of gold, arranged in the cloſeſt manner poſſible, having their centers in one plane, compoſe a cruſt, in which, $\frac{13}{43}$ds, or ſomewhat more than $\frac{12}{40}$ths, of its bulk is matter. Therefore, the denſity of ſuch a cruſt is ſomewhat greater than 12 times that of water, ſince $\frac{1}{40}$th only of the bulk of water is ſuppoſed to be matter, and $\frac{39}{40}$ths is pore.

S. Horſley.

The firſt of theſe two Problems, enabled me to determine the greateſt poſſible number of ſpherical particles of a given magnitude, that could find room to lie at one time upon the ſurface of the Sun; and, by the ſecond, I found the denſity of the cruſt, which ſuch particles, in the cloſeſt arrangement poſſible, with a given denſity of each particle ſeparately, would compoſe.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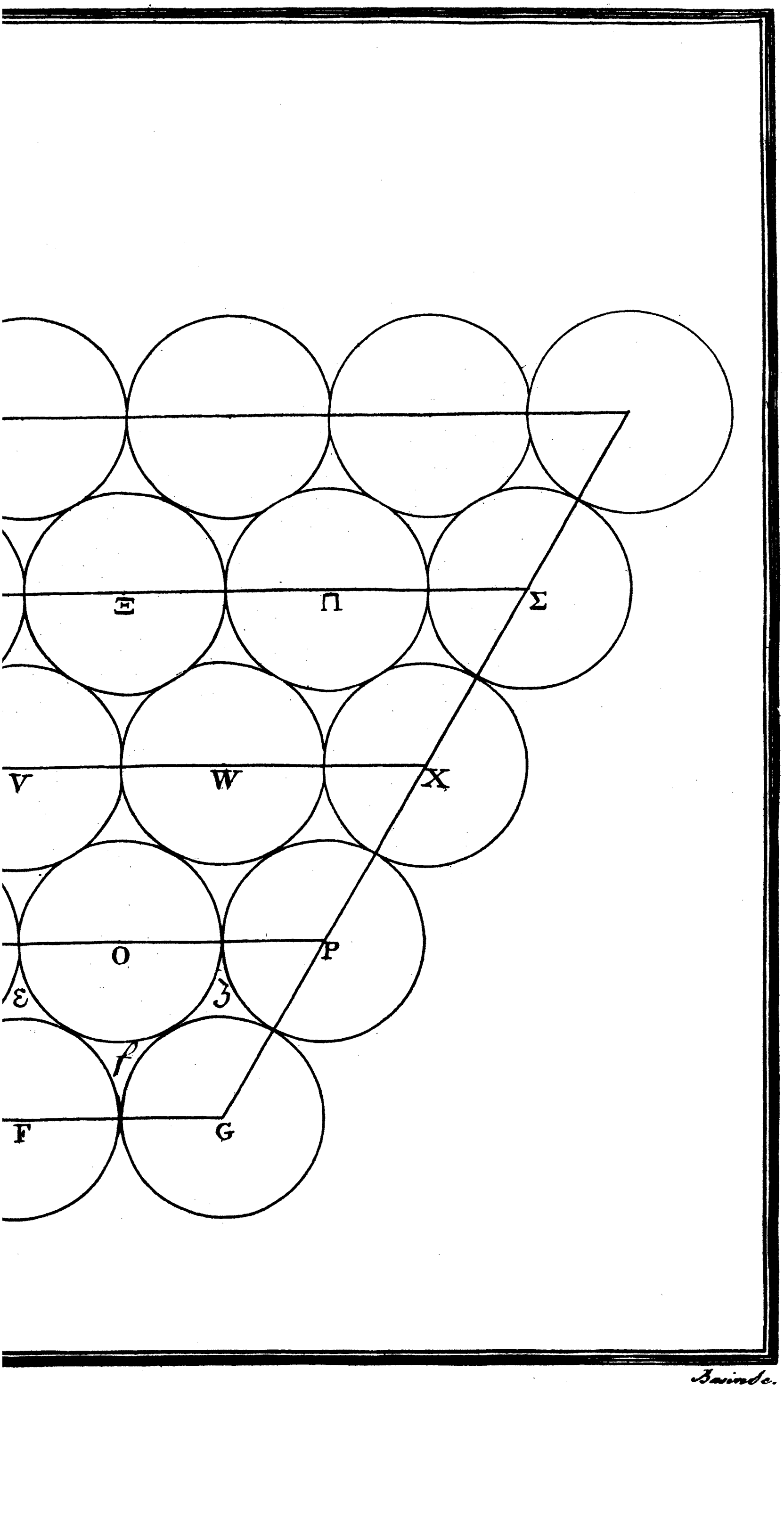

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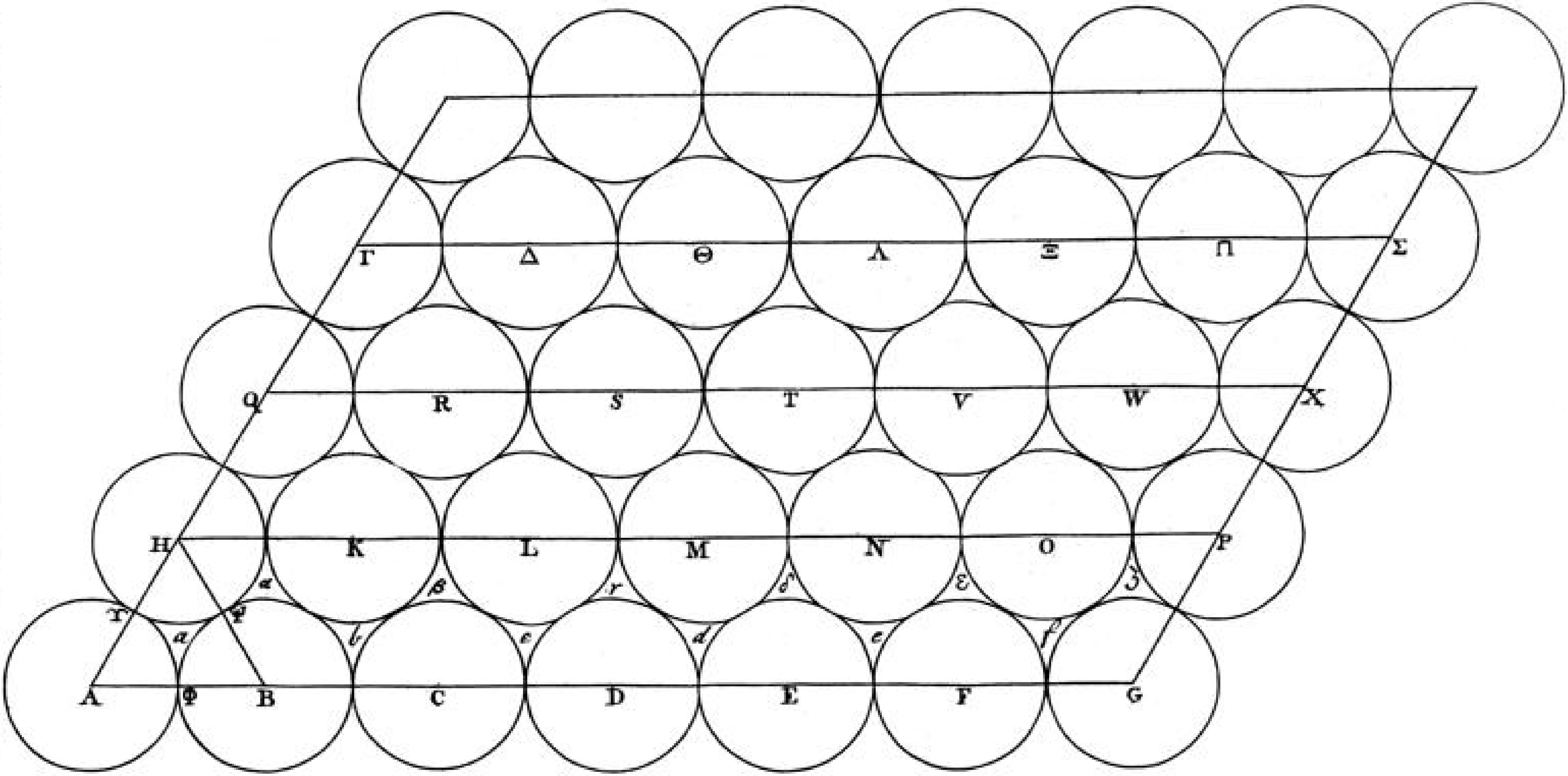

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